Pre-class Warm-up

With reference to the picture on the right, what is the sum

$$\sum_{i=1}^{5} f(c_i)\cdot(x_i - x_{i-1})$$

called?

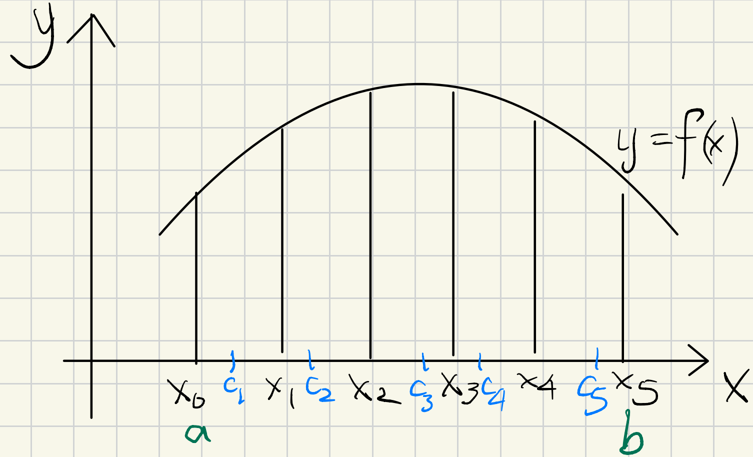
a. a Cauchy sum

b. a Newton sum

c. a Dedekind sum

d. a Riemann sum

e. Some other kind of sum



$$\int_a^b f(x)\,dx = \lim_{\substack{x_i - x_{i-1} \\ \text{gets small}}} \sum f(c_i)(x_i - x_{i-1})$$

Midterm Exam 1 on Tuesday next week is in your discussion session.

Sections 5.1 and 5.2: double integrals over rectangles

We learn

- Different notations for the double integral
- Interpretation as volume under the graph
- Interpretation as volume swept out by a slice (Cavalieri's principle)
- Proper definition using Riemann sums
- Some theoretical things: continuous implies integrable, bounded with restrict discontinuities implies integrable
- Fubini's theorem
- How to calculate integrals

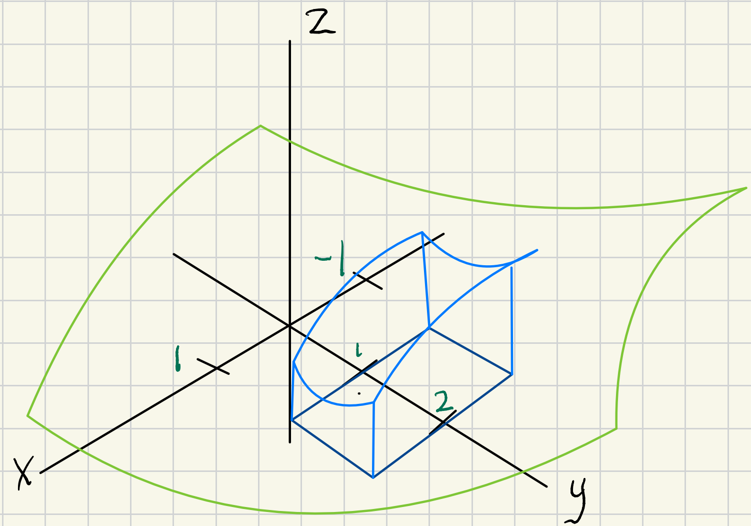
Examples: a. Find $\int_1^2 \left(\int_{-1}^1 x^2y + y^3 \, dx \right) dy$

b. Find $\iint_R x^2y + y^3 \, dA$

where R is the rectangle [-1,1] x [1,2]

$= \{(x,y) \mid x \in [-1,1],\ y \in [1,2]\}$

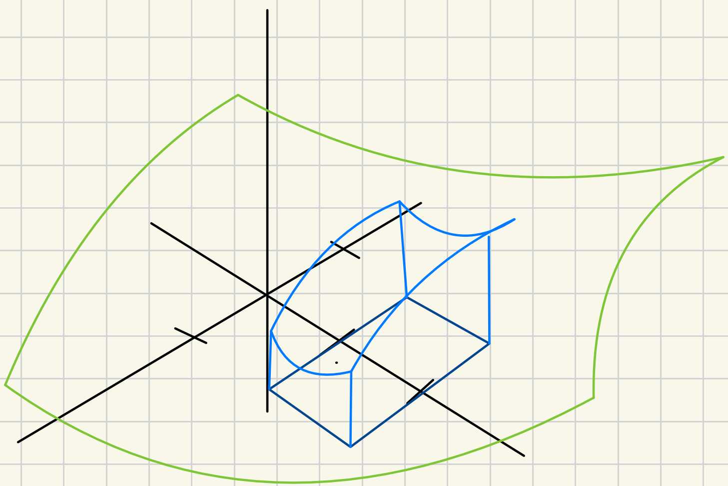
c. Find the volume under the graph of f(x,y) = x^2y + y^3 above the rectangle [-1,1] x [1,2]

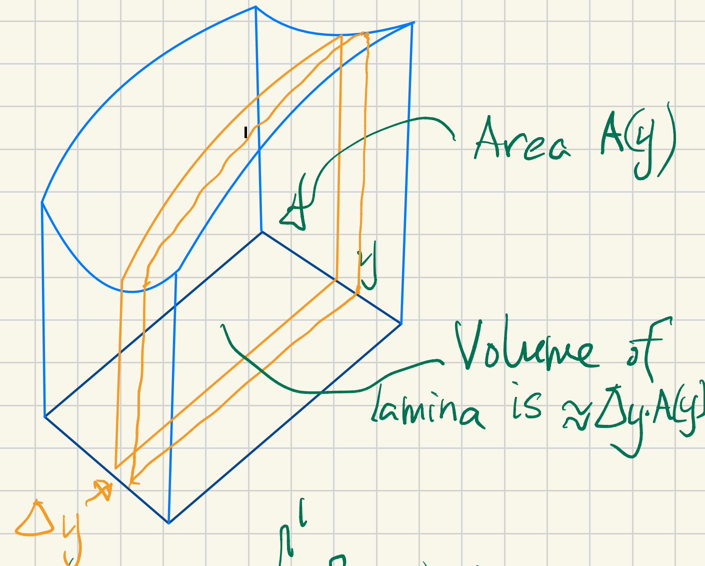


Examples: a. Find $\int_1^2 \left(\int_{-1}^{1} x^2y + y^3 \, dx \right) dy$

$$= \int_1^2 \left[\frac{x^3}{3}y + xy^3 \right]_{-1}^{1} = \int_1^2 \frac{2}{3}y + 2y^3 \, dy$$

$$= \left[\frac{y^2}{3} + \frac{1}{2}y^4 \right]_1^2 = \frac{3}{3} + \frac{15}{2} = 8\frac{1}{2}$$

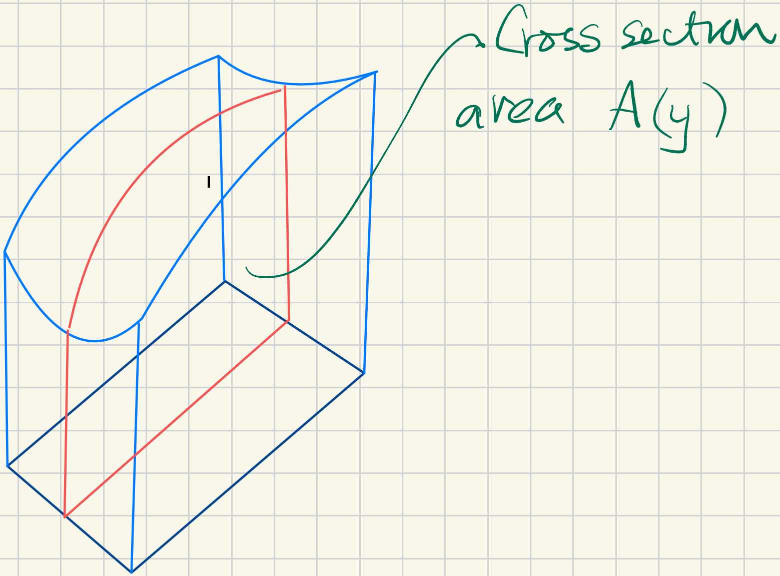




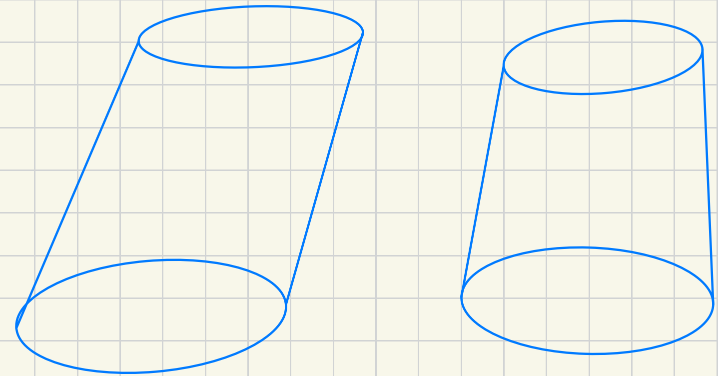
For each y, $\int_{-1}^{1} f(x,y) \, dx$ = area of orange slice = $A(y)$

Now $\int_1^2 A(y) \, dy$ is the volume of blue piece.

Cavalieri's Principle



The volume of a solid is the integral of its cross-section area with respect some coordinate pointing out of the cross-section.



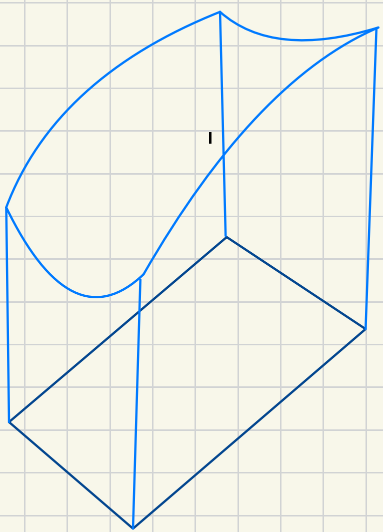
If the horizontal cross-sections have the same area, the two have the same volume.

Examples: a. Find $\int_{-1}^{1}\left(\int_{1}^{2} x^2y + y^3 \, dy\right)dx$

$$= \int_{-1}^{1}\left[\frac{x^2y^2}{2} + \frac{y^4}{4}\right]_1^2 dx = \int_{-1}^{1}\left(\frac{3x^2}{2} + \frac{15}{4}\right)dx$$

$$= \left[\frac{x^3}{2} + \frac{15}{4}x\right]_{-1}^{1} = \frac{1}{2} + \frac{1}{2} + \frac{15}{4} + \frac{15}{4}$$

$$= 8\tfrac{1}{2}$$



Informal Fubini's theorem.

$$\int\int f \, dx\, dy = \int\int f \, dy\, dx.$$

What's wrong with this?
We don't yet know what we mean by the volume under the graph.
We don't have a proper definition of the integral.

Question:

What is $\int_1^2 \int_{-1}^1 dx\,dy$?

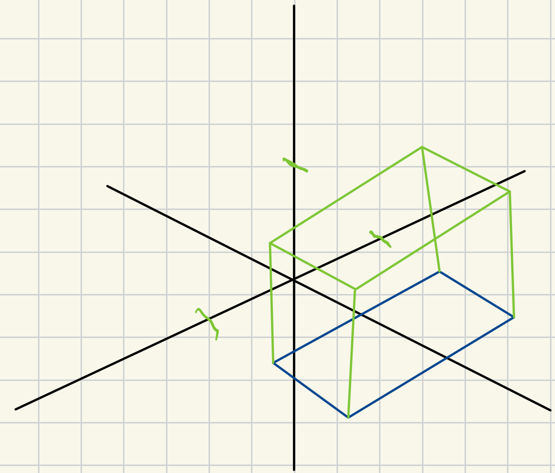
a. 0

b. 1

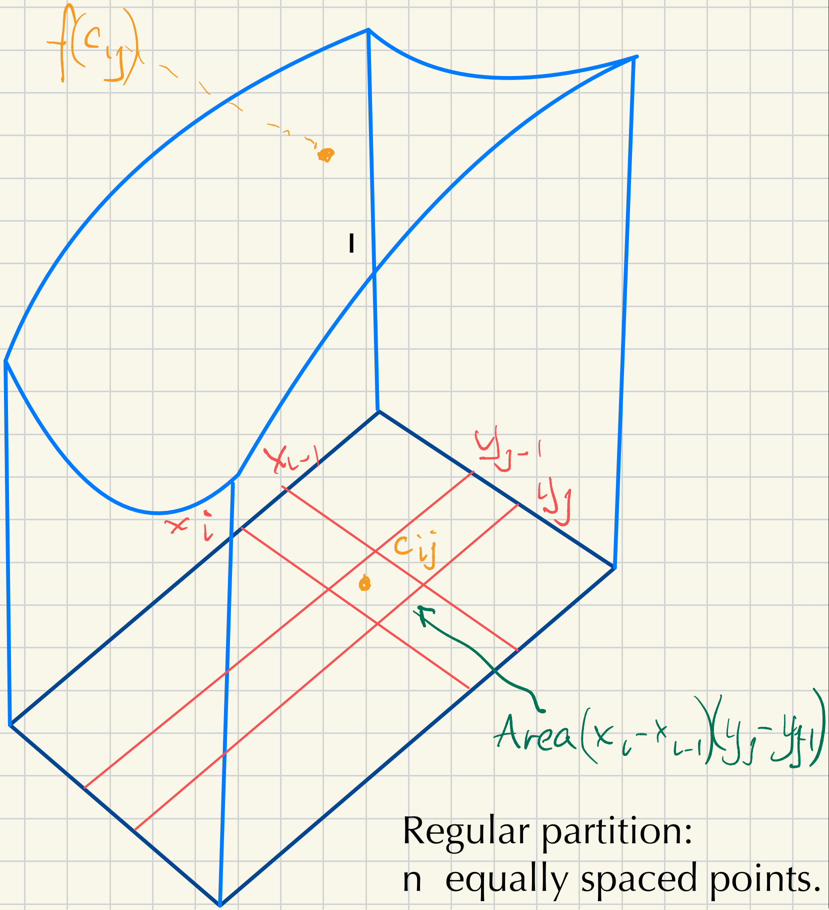
c. 2 ✓

d. 4

e. 1/2



Riemann sums



Regular partition:
n equally spaced points.

$$\sum f(c_{ij})(x_i - x_{i-1})(y_j - y_{j-1})$$

is a Riemann sum.

We say the function $f(x,y)$ is integrable if

these Riemann sums $\longrightarrow d$

for some fixed d

as $(x_i - x_{i-1}), (y_j - y_{j-1}) \longrightarrow 0$

and for arbitrary c_{ij}.

d is the value of the integral.

What we do using Riemann sums

- We get a definition of the integral that does not depend on the order in which we do x and y.
- We get a proper definition of volume under the graph.
- We show that continuous functions are integrable.
- We show that continuous functions apart from discontinuities that lie on curves that are the graphs of functions are integrable.
- We prove Fubini's theorem
- We establish formal properties of the integral